Rational Expectations of Inflation Undergoing Unforeseeable Change*

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Abstract

The rational expectations hypothesis (REH) is widely regarded as representing rational market participants' expectations. However, REH presumes that market participants have perfect probabilistic foresight of future outcomes. In real-world markets, nonrepetitive, and thus unforeseeable, structural change renders such foresight inherently impossible. This paper proposes a new theoretical approach, which we call the Knight-Muth hypothesis (KMH), to representing rational participants' expectations as consistent with inflation undergoing unforeseeable structural change. Because such change gives rise to Knightian uncertainty, KMH recognizes that even when rational participants have access to full information, it is impossible for them to have perfect probabilistic foresight of future inflation. In contrast to REH, this enables KMH to reconcile key features of survey-based expectations with the assumption that participants are rational and have access to full information. Behavioral models and limited-information REH can account for only some of these key features by assuming participants are irrational or lack information. KMH implies that rational participants with full information form expectations that deviate from ex post inflation in nonrepetitive ways. Consistent with empirical findings, this implies nonrepetitive structural shifts in the bias and correlation of ex post forecast errors with ex ante information. Also consistent with survey-based empirical findings, KMH recognizes that rational participants form diverse expectations, even when they base them on full information, and that psychological factors play an important role in their expectations formation, particularly during periods of unforeseeable structural change in the inflation process.

Keywords: Inflation Expectations; Structural Change; Unforeseeable Change; Knightian Uncertainty; Muth's Hypothesis;

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1 Introduction

This paper rests on the observation that the economy undergoes nonrepetitive structural changes whose consequences and timing cannot be fully understood in advance. Major structural changes are engendered not only by economic policy shifts, geopolitical conflict, pandemics, and various other factors, but also by modern market economies' essential feature: incentivizing profit-seeking participants to seek new ways to allocate their resources. Ultimately, participants' innovation changes the economy's structure in ways that are not repetitive of past changes. This nonrepetitiveness renders the structural change unforeseeable: no one can objectively know, even in probabilistic terms, when and how the economy's structure will change and thus how structural change will affect future economic outcomes.

The rational expectations hypothesis (REH) has been widely considered the way to represent rational market participants' expectations in macroeconomic models. By assuming constant parameters, most REH models abstract from changes in the economy's structure altogether. Those REH models that allow for structural change typically formalize it as Markov switching between a small set of repetitive regimes, so that future regimes are assumed to be identical to previous regimes. In such a model, REH hypothesizes that participants' expectations can be represented with the model's "objective" conditional expectation of future outcomes. Muth (1961, p. 316) motivated this hypothesis by assuming that "expectations of firms (or, more generally, the subjective probability distribution of outcomes) tend to be distributed, for the same information set, about the prediction of the theory (or the 'objective' probability distribution of outcomes)." Thus, although REH is boldly abstract, it presumes that participants can foresee, in probabilistic terms, exactly how future outcomes will unfold. As Lucas (2005, p. 283) put it, "the assumption of (REH) reduces to perfect foresight."

However, even if it occurs only infrequently, unforeseeable change in real-world markets renders such knowledge of the future inherently unattainable. Thus, although REH models specify participants' expectations as consistent with the models' predictions, we argue that they do not represent expectations of rational participants in real-world markets. Modeling these expectations requires that economists formalize structural change as unforeseeable and specify participants' expectations as consistent with the Knightian uncertainty arising from such change. To this end, we present a tractable and empirically testable theoretical approach, which we call the Knight-Muth hypothesis (KMH).

KMH formalizes Knight's (1921) conjecture that understanding real-world market outcomes requires acknowledging that participants face Knightian uncertainty arising from unforeseeable structural change. It does so by assuming that its models' "objective" parameters shift intermittently between nonrepetitive values that are bound only to lie within an interval. Although the past parameter values can be estimated based on historical time-series data, the model's future parameters are inherently unknowable and cannot be characterized even probabilistically. This renders the model's objective probability distribution of future outcomes inherently unknowable ex ante. However, the boundedness of the objective parameters implies that a set of probability distributions constitutes the model's predictions of future outcomes. This set of probability distributions formalizes Knightian uncertainty: the uncertainty about future outcomes cannot be reduced ex ante to a single probability distribution.

KMH provides a new implementation of Muth's (1961) hypothesis that an economist can acknowledge market participants' rationality by specifying their expectations as consistent with his or her model's predictions of future outcomes. REH implements this hypothesis in a model with no or foreseeable structural change. Representing participants' expectations as consistent with the predictions of such models presumes that participants face only probabilistic risk and have perfect probabilistic foresight of future outcomes. In contrast, because KMH implements Muth's hypothesis in models recognizing unforeseeable change, it acknowledges that market participants face Knightian uncertainty and thus cannot have such perfect probabilistic foresight.

KMH represents participants' expectations with a model-consistent subjective expectation. We define this expectation as the model's "objective" conditional expectation of future outcomes, but where the unknowable future objective parameters are replaced by subjective parameters. These subjective parameters represent participants' inherently imperfect ex ante assessment of the future objective parameters.

Over time, we assume that the subjective parameters undergo nonrepetitive, and thus unforeseeable, shifts as participants revise their assessments, and thus their expectations, in anticipating and responding to the unforeseeable change in the objective parameters. Importantly, however, KMH leaves the subjective parameters, and how they are revised, exogenous, and constrains them only to lie within the same interval as the objective parameters. This is sufficient to render the representation of participants' expectations consistent with the model: they lie within the set of conditional expectations that constitute the model's predictions of future outcomes.

A vast empirical literature has explored full-information REH (FIRE) models' difficulties accounting for survey-based expectations of inflation and other macroeconomic variables. In contrast to constant-parameter FIRE models' prediction that participants' forecast errors should be unpredictable, survey-based ex post forecast errors have been found to be biased and correlated with ex ante information.[1] Survey-based expectations also show that participants "exhibit substantial disagreement in their expectations about macroeconomic outcomes" (Andre et al., 2022, p. 2959). Because REH models permit

[1]See Coibion, Gorodnichenko, and Kamdar (2018) for an overview in the context of inflation expectations and Coibion and Gorodnichenko (2015), Bordalo et al. (2020), Angeletos, Huo, and Sastry (2021), Kohlhas and Walther (2021), Hajdini and Kurmann (2024), and Frydman and Tabor (2024).

only one model-consistent expectation given the available information, they can account for this diversity of participants' expectations as arising only from differences in the information available to participants. Finally, although behavioral economists have provided important empirical and experimental evidence that psychological factors influence participants' expectations, such factors play no role in REH models.

The literature has accounted for FIRE's empirical difficulties mainly by assuming that rational participants have access only to limited information (Coibion and Gorodnichenko, 2015), or that participants are irrational, as psychological factors distort their expectations relative to FIRE (Bordalo et al., 2020).

We advance a different explanation: FIRE's empirical difficulties do not arise because participants face limited information or are irrational, but because REH presumes participants have perfect probabilistic foresight of the future. Unforeseeable change in real-world markets renders such foresight inherently unattainable, even when participants are rational and have access to full information. Thus, REH models do not represent the expectations of rational participants in real-world markets, irrespective of whether participants are assumed to have access to full or limited information.

Because KMH acknowledges that market participants cannot have perfect probabilistic foresight, it can reconcile market participants' rationality with the nonrepetitive systematic forecast errors that are a key feature of survey-based expectations. KMH implies that the bias and correlation of participants' ex post inflation forecast errors with ex ante information undergo nonrepetitive structural shifts between values that can be positive, negative, and zero during different periods. In support of this prediction, Frydman and Tabor (2024) present empirical evidence that the parameters of two widely used survey-based forecast-error regressions have undergone nonrepetitive structural shifts.

KMH acknowledges that there are many rational ways to use the same information in forming expectations of future outcomes. Thus, we argue that the diversity of participants' survey-based expectations arises from participants facing unforeseeable change and the Knightian uncertainty engendered by such change. Indeed, Andre et al. (2022) provide compelling evidence that participants' expectations are diverse even when they have access to the same information. We show how this diversity can be formalized in a KMH model by representing participants' model-consistent expectations in terms of different subjective parameters.

KMH also formalizes Keynes' (1936, pp. 162-163) important insight that when the future is inherently unknowable, psychological factors, such as market sentiment and narratives, play an essential role in rational participants' expectation formation and decision-making. We show how the influence of psychological factors on participants' subjective parameters, and thereby their model-consistent expectations, can be formalized in a KMH model.

The paper is structured as follows. In Section 2, we present KMH in a baseline New

Keynesian Phillips curve (NKPC) model of inflation and compare it to REH. We show how KMH formalizes structural change in the parameter representing the pass-through of the output gap to inflation as unforeseeable and how KMH represents participants' inflation expectations as consistent with Knightian uncertainty about future inflation. We also explain how KMH enables an economist to formalize both the influence of psychological factors on participants' model-consistent expectations and Phelps' (1970) conjecture that participants' expectations play an autonomous role in driving inflation, and how the model implies nonrepetitive systematic forecast errors. In Section 3, we illustrate how KMH can formalize the diversity of rational participants' inflation expectations. To do so, we modify the baseline NKPC model in Section 2 by assuming that participants have diverse, model-consistent inflation expectations. We also show that although the baseline NKPC model in Section 2 models the aggregate of participants' inflation expectations, this formalization is consistent with the underlying diversity of participants' inflation expectations. Section 4 concludes with a brief overview of recent findings of structural change in inflation dynamics and highlights KMH's promise in developing a theoretical account of these findings.

2 The Knight-Muth Hypothesis in a New Keynesian Phillips Curve Model for Inflation

This section presents our approach to opening macroeconomic models to unforeseeable change and representing the expectations of rational market participants facing Knightian uncertainty arising from such change. As it builds on the ideas of Knight (1921) and Muth (1961), we call this approach the Knight-Muth hypothesis (KMH).

To highlight KMH's main features, we present it in a baseline New Keynesian Phillips curve (NKPC) model for inflation where we assume that only the parameter representing the pass-through of the output gap to inflation undergoes unforeseeable change. The baseline NKPC relates inflation, π_t, to the aggregate of market participants' expectation of the next period's inflation, $\mathrm{F}_t(\pi_{t+1})$, and an exogenous output gap (or marginal costs), x_t, as given by:

$$\pi_t = \beta\, \mathrm{F}_t\left(\pi_{t+1}\right) + \kappa_t x_t, \tag{1}$$

for $t = 1, 2, \ldots$, where we assume that the parameter $0 < \beta < 1$ is constant and the parameter κ_t, representing the pass-through of the output gap to inflation at time t, shifts over time. We assume that the output gap follows the stationary mean-zero autoregressive process with constant parameters given by:

$$x_t = \rho x_{t-1} + \epsilon_t, \tag{2}$$

for $t = 1, 2, \ldots$ and where $0 < \rho < 1$ is a constant parameter, $\epsilon_t \sim iidN(0, \sigma^2)$, and the initial value, x_0, is given. Adopting Muth's (1961, p. 316) nomenclature, we refer to (1) and (2) as the model's "objective" inflation process, and its parameters, $(\beta, \kappa_t, \rho, \sigma^2)$, as the "objective" parameters. Our illustration of KMH focuses on structural change in the objective parameter κ_t.

In the following subsections, we explain, first, how KMH formalizes structural change in κ_t as unforeseeable and how KMH implements Muth's hypothesis to represent participants' inflation expectations, $\mathrm{F}_t(\pi_{t+1})$ in (1), as consistent with the model's predictions of future inflation. Second, we show how KMH reconciles participants' rationality with the influence of psychological factors on their inflation expectations; how KMH accords participants' expectations an autonomous role in driving inflation; and how KMH implies that rational participants make nonrepetitive forecast errors.

2.1 Formalizing Unforeseeable Structural Change

KMH builds on Knight's conjecture that to understand profit-seeking activity in real-world markets, economists must recognize the importance of structural change that cannot "by any method be reduced to an objective, quantitatively determinate probability" (Knight, 1921, pp. 231-232) together with the "true"—nowadays called Knightian—uncertainty resulting from such change. As he put it: "if all changes were to take place in accordance with invariable and universally known laws, [so that] they could be foreseen for an indefinite period in advance of their occurrence (...) profit or loss would not arise" (Knight, 1921, p. 198).

To be sure, if the economy's structure changed in unforeseeable ways at all times, so that there were no regularities, formal models would be useless. At the other extreme, most REH models abstract from structural change altogether by assuming that their parameters remain constant.

KMH assumes an intermediate position. It does so by hypothesizing that inflation can be characterized with a stochastic process that undergoes nonrepetitive structural changes only intermittently, and that the periods between these changes are stable enough to be modeled as constant. Thus, we assume that the objective parameter of the inflation process, κ_t, intermittently shifts between nonrepetitive regimes. Moreover, we assume the regularity that the pass-through of the output gap to inflation is positive and bounded in all regimes. These assumptions enable us to provide a tractable and empirically testable formalization of Knight's conjecture that the economy's structure changes in ways and at times that cannot be foreseen, even in probabilistic terms.

We formalize these assumptions as follows,

$$\kappa_t = \kappa^{(i)} \in I^{\kappa} = [\kappa_L, \kappa_U], \quad t \in [\tau_{i-1}, \tau_i - 1], \quad i = 1, 2, \ldots \tag{3}$$

where $\tau_i > \tau_{i-1}$ for all i, $\tau_0 = 1$, and the bounds of the interval I^k satisfy $0 < \kappa_L < \kappa_U$. The first part of (3) formalizes κ_t to shift only intermittently between nonrepetitive regimes, $\{\kappa^{(1)}, \kappa^{(2)}, \ldots\}$, at times $\{\tau_1, \tau_2, \ldots\}$. The second part of (3) formalizes the regularity that the pass-through of the output gap to inflation is always positive by restricting $\kappa^{(i)}$ to lie within the positive interval I^κ during each regime.

Given a sample of historical time-series data, $\{\pi_t, x_t\}_{t=1}^T$, this formalization of structural change in the inflation process can be tested empirically using standard statistical tools. This can be done by estimating (1) using survey forecasts to measure the market's inflation expectations or by estimating the reduced-form expression for inflation that we derive below. It requires, however, an econometric approach that identifies from the data the number and timings of the structural shifts over the sample period, $\{\tau_i\}_{i=1}^N$ for some $N > 0$.[2] The range of estimates in past regimes enables empirical assessment of the bounds of the interval I^κ, which can be combined with theoretical constraints, such as the positivity of the interval.

Although the values of κ_t in past regimes can be empirically estimated, the formalization of the structural change as nonrepetitive renders future structural changes in the inflation process unforeseeable. Because the values of the objective parameter κ_t in future regimes differ from those in past regimes, those future values cannot be estimated based on historical time-series data. Thus, at any time t, there can be no objective assessment of the values of κ_{t+h} for $h > 0$, which cannot even be characterized probabilistically. Such unforeseeable change gives rise to Knightian uncertainty about future inflation, in addition to the probabilistic risk represented in the model by future shocks to the output gap.

In contrast, most REH models abstract from structural change altogether by restricting their parameters to remain constant, $\kappa_t = \kappa$ for all t. Those REH models that allow for structural change typically formalize it as Markov switching between repetitive regimes, thereby rendering the structural change foreseeable in probabilistic terms. For example, with two regimes, $\kappa_t = \kappa(s_t) = \kappa^{(i)}$ when $s_t = i$ for $i \in \{1, 2\}$ and the switching probabilities, $p_{i,j} = \mathrm{P}(s_{t+1} = j | s_t = i)$, are given by $p_{1,1} = p_1$, $p_{1,2} = 1 - p_1$, $p_{2,1} = 1 - p_2$, and $p_{2,2} = p_2$. Because the parameters and switching probabilities in future regimes are assumed to be identical to those in the past, their values can be estimated based on historical time-series data. Thus, future structural changes can be foreseen in probabilistic terms: if the current regime is $s_t = i$, it can be foreseen that the next period's objective parameter $\kappa(s_{t+1})$ will take the value $\kappa^{(1)}$ with probability $p_{i,1}$ and the value $\kappa^{(2)}$ with

[2] For example, the timing of structural shifts can be identified using the standard test by Bai and Perron (1998, 2003a, 2003b) or an indicator-saturation approach with the Autometrics model-selection algorithm, see Doornik (2009), Castle, Doornik, and Hendry (2012), and Castle et al. (2015). Frydman and Tabor (2024) show how indicator-saturation with Autometrics can be used to identify the timing and estimate the magnitude of structural shifts in regressions of survey-based ex post inflation forecast errors on ex ante information.

probability $p_{i,2}$. Although it cannot be known ex ante which of the two possible values $\kappa(s_{t+1})$ will take, it can be characterized probabilistically with the conditional expectation $\mathrm{E}(\kappa(s_{t+1})|s_t = i) = p_{i,1}\kappa^{(1)} + p_{i,2}\kappa^{(2)}$. Consequently, the only uncertainty about future inflation would be probabilistic risk about the model's future latent states and random shocks to the output gap.

We emphasize that it is the assumption that κ_t shifts between nonrepetitive regimes—not whether the timing of structural shifts is specified as deterministic, as in (3), or stochastic—that distinguishes the KMH model's specification of structural change as unforeseeable from REH's formalization of structural change as foreseeable. Indeed, as an alternative to (3), we could, for example, assume that κ_t shifts stochastically between nonrepetitive regimes according to a Markov chain with nonrepetitive states. In that case, κ_t would depend on a latent state, $s_t \in \{1, 2, \ldots\}$, such that $\kappa_t = \kappa(s_t) = \kappa^{(i)}$ when $s_t = i$ for $i = 1, 2, \ldots$, and where the shifting probabilities, $p_{i,j} = \mathrm{P}(s_{t+1} = j|s_t = i)$, are given by $p_{i,i} = p_i$, $p_{i,i+1} = 1 - p_i$, and $p_{i,j} = 0$ for $i = 1, 2, \ldots$ and $j \notin \{i, i+1\}$. Thus, in the regime i, the process will stay in the current regime with probability p_i or shift to the next regime with probability $1 - p_i$, but it will never return to past regimes or shift beyond the next regime. This formalization would also render the structural change unforeseeable: the parameters and switching probabilities in future regimes differ from those in past regimes and thus cannot be objectively assessed or characterized probabilistically ex ante.

2.2 Rational Expectations of Inflation Under Knightian Uncertainty

REH implements Muth's (1961) hypothesis in an inflation model that assumes no or foreseeable structural change by specifying participants' inflation expectations with the model's objective conditional expectation of future inflation. This simultaneous specification of the inflation process and participants' inflation expectations in an intertemporal model, such as (1), renders participants' expectations consistent with the model's predictions: participants' expectations are identical to the conditional expectation of the resulting reduced-form expression for future inflation. In this way, REH formalizes Muth's (1961, p. 316) conjecture that participants' "subjective probability distribution of outcomes (...) tend[s] to be distributed, for the same information set, about the (...) 'objective' probability distribution of outcomes." Thus, REH presumes that participants have perfect probabilistic foresight of future outcomes.

In the 1970s, Robert Lucas advanced compelling arguments that persuaded most macroeconomists to rely on REH to represent participants' expectations. As he recounted in his Nobel lecture, intertemporal models that do not conform to Muth's hypothesis suffer from a "glaring" (Lucas, 1995, p. 255) inconsistency between "actual equilibrium prices and (...) the price expectations that the theory imputed to individual agents" (p. 254). Indeed, as Lucas emphasized, specifying participants' expectations as model-consistent is an essential condition of coherent macroeconomic model building. However, for such specifications to represent expectations of rational participants in real-world markets, the economist's model should recognize that outcomes in such markets undergo unforeseeable change.

To this end, KMH proposes a new implementation of Muth's hypothesis to represent the inflation expectations of rational participants facing Knightian uncertainty arising from unforeseeable change in the inflation process.[3] The crucial implication of formalizing structural change as unforeseeable is that the future objective parameters, κ_{t+h} for $h > 0$, cannot be objectively assessed or even characterized probabilistically ex ante. KMH thereby acknowledges that market participants cannot have perfect probabilistic foresight of future inflation: the model's objective distribution of future inflation is inherently unknowable when participants form their expectations. However, the boundedness of the objective parameters implies that a set of conditional expectations formalizes Knightian uncertainty and constitutes the model's predictions of future inflation. KMH implements Muth's hypothesis under Knightian uncertainty by specifying participants' inflation expectation with a subjective conditional expectation that is consistent with the model's predictions in the sense that it lies within this set.

2.2.1 A Model-Consistent Representation of Participants' Subjective Inflation Expectations

KMH hypothesizes that market participants' inflation expectations, $\mathrm{F}_t(\pi_{t+1})$ in (1), can be represented with a model-consistent subjective conditional expectation. We define this by the conditional expectation of π_{t+1} implied by the inflation process in (1) and (2), but where the unknowable future objective parameters, κ_{t+h} for $h = 1, 2, \ldots$, are replaced by a subjective parameter, which we denote by $\bar{\kappa}_t$.[4] The subjective parameter represents participants' assessment, at time t, of the future objective parameters.[5]

We constrain the subjective parameter $\bar{\kappa}_t$ to lie within the same interval as the objec-

[3]Frydman and Goldberg (2007, 2011) argued that macroeconomic and finance models should recognize that change is unforeseeable. They conjectured that Muth's hypothesis would play a central role in representing expectations in intertemporal models that are open to such change (Frydman and Goldberg, 2015). KMH formalizes and extends these ideas and enables the derivation of their theoretical implications and empirical predictions.

[4]Here, we make the simplifying assumption that participants' subjective assessment is identical for all forecast horizons $h > 0$ at time t. In Appendix B, however, we derive the model-consistent representation of participants' inflation expectations for the general case where we allow the subjective assessment to shift over the forecast horizon.

[5]Strictly, within the model, the subjective parameter $\bar{\kappa}_t$ represents participants' subjective assessment at time t of the objective parameter at time $t + 1$, κ_{t+1}, as well as participants' subjective assessment at time t of their subjective assessment at time $t + 1$ of κ_{t+2} and additional higher-order subjective assessments.

tive parameter at all times. We show this is sufficient to render participants' subjective expectations consistent with the model's predictions about future inflation.

Over time, we assume that $\bar{\kappa}_t$ intermittently shifts between nonrepetitive regimes as participants revise their inflation expectations in anticipation of and response to the unforeseeable change in the objective parameter, κ_t, of the inflation process. Similarly to the formalization of unforeseeable change in κ_t in (3), we formalize this as follows,

$$\bar{\kappa}_t = \bar{\kappa}^{(j)} \in I^{\kappa}, \qquad t \in [\bar{\tau}_{j-1}, \bar{\tau}_j - 1], \qquad j = 1, 2, \ldots, \tag{4}$$

where $\bar{\tau}^j > \bar{\tau}_{j-1}$ for all j, $\bar{\tau}_0 = 1$, and I^{κ} is defined in (3). The first part of (4) constrains $\bar{\kappa}_t$ to shift only intermittently between nonrepetitive values, $\{\bar{\kappa}^{(1)}, \bar{\kappa}^{(2)}, \ldots\}$, at times $\{\bar{\tau}_1, \bar{\tau}_2, \ldots\}$. Thus, we assume that between the intermittent shifts, participants' subjective parameter is sufficiently stable to be modeled as constant. The second part of (4) constrains $\bar{\kappa}^{(j)}$ to lie within the objective parameters' positive interval, I^{κ}, at all times, so that participants always assess the future direct impact of the output gap on inflation to be positive.

Let $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_t)|x_t)$ denote participants' subjective conditional expectation, at time t, of the next period's inflation, given the subjective parameter, $\bar{\kappa}_t$, and conditional on the information x_t. We define this as the conditional expectation of the subjective process for future inflation given by (1) and (2), but where the unknown future objective parameters, κ_{t+h} for $h = 1, 2, \ldots$, are replaced by the subjective parameter, $\bar{\kappa}_t$, and $\mathrm{F}_{t+h}(\pi_{t+h+1}) = \mathrm{E}(\pi_{t+h+1}(\bar{\kappa}_t)|x_{t+h})$ for all $h = 1, 2, \ldots$. Thus, participants' inflation expectations are based on their subjective process for $\pi_{t+h}(\bar{\kappa}_t)$, for a fixed t and $h = 1, 2, \ldots$, given by:

$$\pi_{t+h}\left(\bar{\kappa}_t\right) = \beta\, \mathrm{E}\left(\pi_{t+h+1}\left(\bar{\kappa}_t\right)|x_{t+h}\right) + \bar{\kappa}_t x_{t+h}. \tag{5}$$

Here, x_{t+h} is specified with the constant-parameter autoregressive process in (2), which implies that $\mathrm{E}(x_{t+h}|x_t) = \rho^h x_t$ for $h > 0$ as $0 < \rho < 1$.[6]

An explicit expression for the model-consistent subjective expectation, $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_t)|x_t)$, can be derived by forward iteration using (5) for $h = 1, 2, \ldots$. The first step of this forward iteration is given by:

$$\begin{aligned} \mathrm{E}(\pi_{t+1}(\bar{\kappa}_t)|x_t) &= \mathrm{E}\left(\beta\, \mathrm{E}(\pi_{t+2}\left(\bar{\kappa}_t\right)|x_{t+1}) + \bar{\kappa}_t x_{t+1}|x_t\right) \\ &= \beta\, \mathrm{E}\left(\pi_{t+2}(\bar{\kappa}_t)|x_t\right) + \bar{\kappa}_t\, \mathrm{E}\left(x_{t+1}|x_t\right). \end{aligned}$$

This is similar to the standard REH solution, except that the forward iteration is performed with respect to the subjective process for future inflation, $\pi_{t+h}(\bar{\kappa}_t)$ in (5) for a fixed t and $h = 1, 2, \ldots$, instead of the objective process in (1). The following proposition

[6] We assume that participants' subjective inflation expectations can be represented in terms of the objective parameters, (β, ρ), as they are assumed to be constant.

states the unique expression for the subjective conditional expectation that follows from this continued forward iteration, given the assumptions $0 < \beta < 1$ and $0 < \rho < 1$.

Proposition 1. *Let* $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_t)|x_t)$ *denote the model-consistent subjective conditional expectation of* $\pi_{t+1}(\bar{\kappa}_t)$*, where* $\bar{\kappa}_t$ *denotes participants' subjective parameter at time* t*, as specified in (4), and* $\pi_{t+h}(\bar{\kappa}_t)$ *denotes participants' subjective process for future inflation, as specified in (5) for fixed* t *and* $h = 1, 2, \ldots$ *with* x_t *following (2). The subjective conditional expectation is uniquely given by:*

$$\mathrm{E}\left(\pi_{t+1}\left(\bar{\kappa}_t\right)|x_t\right) = \sum_{h=1}^{\infty} \beta^{h-1} \bar{\kappa}_t \,\mathrm{E}\left(x_{t+h}|x_t\right) = \frac{\rho \bar{\kappa}_t}{1-\beta\rho} x_t. \tag{6}$$

The proof of the proposition is presented in Appendix A.

KMH hypothesizes that market participants' inflation expectation in (1) can be represented with the model-consistent subjective expectation in (6): $\mathrm{F}_t(\pi_{t+1}) = \mathrm{E}(\pi_{t+1}(\bar{\kappa}_t)|x_t) = \rho\bar{\kappa}_t(1-\beta\rho)^{-1}x_t$. This represents participants' subjective expectation of the next period's inflation given their subjective assessment of how future inflation will unfold. It is given by the discounted sum of their subjective expectations of how future output gaps will pass through to inflation.

The functional form and relevant information of KMH's model-consistent representation of participants' inflation expectations are completely determined by the model's specification of the inflation and output gap processes in (1) and (2). Thereby, KMH maintains Muth's assertion that "[t]he way expectations are formed depends specifically on the structure of the relevant system describing the economy" (Muth, 1961, p. 316).

However, because KMH recognizes that participants face Knightian uncertainty, and thus cannot have perfect foresight, it represents their model-consistent inflation expectations in terms of exogenous subjective parameters that generally differ from the future objective parameters of the inflation process. This is the main difference between KMH's and REH's model-consistent representations of participants' expectations.

By leaving the subjective parameters exogenous and only bound to lie within the objective parameters' interval, KMH acknowledges that an economist cannot know exactly how participants form their subjective assessment of future objective parameters, and thus their inflation expectations, when the inflation process undergoes unforeseeable change. Moreover, by allowing the subjective parameters to shift in nonrepetitive ways, KMH acknowledges that neither an economist nor market participants can know in advance how and when participants will revise their inflation expectations in anticipation and response to future unforeseeable changes in the inflation process.

KMH's representation nests the possibility that participants' subjective parameter can be lower, higher, or equal to the current and future objective parameters during different periods. Although KMH does not specify a direct link, we expect the subjective parameter to be close or identical to the objective parameter when the latter has remained constant for a long period and participants assess that no changes will occur in the near future. In contrast, we expect the difference to be large during periods around shifts in the objective parameters. Moreover, the subjective parameter can shift before, after, or at the same time as a shift in the objective parameter. Some shifts might be unanticipated by participants, so that their subjective parameter deviates from the objective parameter and shifts in response with some lag. However, other shifts in the objective parameter might be anticipated by participants, so that a shift in the subjective parameter precedes them. This could happen when, say, a shift in monetary policy is announced before its implementation, causing participants immediately to revise their inflation expectations in response to the announcement. Thus, although we refer to the model's structural change as unforeseeable, this does not imply, and KMH does not assume, that participants can never correctly foresee the magnitude and/or timing of structural shifts in the inflation process. Rather, it implies that we can neither assume that they always do so, nor know in advance whether that will be the case.

In contrast to KMH, REH's model-consistent representation of participants' inflation expectations, including its parameters, is completely determined by its specification of the inflation and output gap processes. For example, in a constant-parameter REH model, $\kappa_t = \kappa$ for all t, the model-consistent inflation expectations are given by $\mathrm{E}(\pi_{t+1}|x_t) = \rho\kappa(1-\beta\rho)^{-1}x_t$. In an REH model with Markov switching between two repetitive regimes, the model-consistent inflation expectations have a solution, given by $\mathrm{E}(\pi_{t+1}(s_{t+1})|x_t, s_t = i) = (p_{i,1}\gamma^{(1)} + p_{i,2}\gamma^{(2)})x_t$, under the conditions stated in Davig and Leeper (2007). Here, the parameters $(\gamma^{(1)}, \gamma^{(2)})$ are completely determined by the objective parameters, $(\kappa^{(1)}, \kappa^{(2)}, p_1, p_2, \beta, \rho)$. This implies that the regime-switching in the inflation process completely determines the regime switches in participants' expectations: the shift in the inflation process immediately leads to a shift in participants' inflation expectations and determines their magnitude.

2.2.2 The Reduced-Form Expression for Inflation

The NKPC in (1) and the model-consistent representation of participants' subjective inflation expectations in (6) implies a reduced-form expression for inflation that depends on both the objective and subjective parameters at time t, which we denote by $\pi_t(\kappa_t, \bar{\kappa}_t)$. It is given by:

$$\pi_t(\kappa_t, \bar{\kappa}_t) = \beta\left(\frac{\rho\bar{\kappa}_t}{1-\beta\rho}x_t\right) + \kappa_t x_t = \left(\frac{\beta\rho}{1-\beta\rho}\bar{\kappa}_t + \kappa_t\right)x_t. \tag{7}$$

This reduced-form expression shows that the relationship between inflation and the output gap is characterized by a sequence of nonrepetitive regimes over time. The regime shifts arise from shifts in the direct pass-through of the output gap to inflation, as formalized by the objective parameter κ_t, as well as from shifts in participants' assessment of future pass-throughs, as formalized by their subjective parameter $\bar{\kappa}_t$.

2.2.3 Consistency of Participants' Expectations with Knightian Uncertainty About Future Inflation

The unforeseeable change in the model's objective and subjective parameters renders future reduced-form inflation inherently unknowable ex ante: the process for future inflation, $\pi_{t+h}(\kappa_{t+h}, \bar{\kappa}_{t+h})$ for $h > 0$, and its conditional expectation are inherently unknowable at time t, as they depend on the unknowable future parameters κ_{t+h} and $\bar{\kappa}_{t+h}$.

Specifically, the conditional expectation of the next period's reduced-form inflation, given κ_{t+1} and $\bar{\kappa}_{t+1}$, is formally given by:

$$\mathrm{E}(\pi_{t+1}(\kappa_{t+1}, \bar{\kappa}_{t+1})|x_t) = \left(\frac{\beta\rho}{1-\beta\rho}\bar{\kappa}_{t+1} + \kappa_{t+1}\right)\rho x_t.$$

However, this conditional expectation does not constitute the model's prediction of the next period's inflation, as it would in an REH model, because the values of κ_{t+1} and $\bar{\kappa}_{t+1}$ are inherently unknowable at time t.

Instead, the boundedness of the unknown future objective and subjective parameters implies that a set of conditional expectations constitutes the KMH model's predictions of future inflation. For the next period's inflation, $\pi_{t+1}(\kappa_{t+1}, \bar{\kappa}_{t+1})$, this is given by the set of conditional expectations defined by all values of κ_{t+1} and $\bar{\kappa}_{t+1}$ within the interval I^{κ}. The following proposition defines this set of predictions and states that the representation of participants' inflation expectations in (6) is consistent with the model's predictions, as it lies within this set.

Proposition 2. *Let $\pi_{t+1}(\kappa_{t+1}, \bar{\kappa}_{t+1})$ denote the model's reduced-form expression for inflation at time $t+1$, given the objective and subjective parameters, κ_{t+1} and $\bar{\kappa}_{t+1}$, as defined in (7). The model's predictions of $\pi_{t+1}(\kappa_{t+1}, \bar{\kappa}_{t+1})$, as viewed from time t, are given by the set of conditional expectations:*

$$\begin{aligned}
&\{\mathrm{E}\left(\pi_{t+1}\left(\kappa_{t+1}, \bar{\kappa}_{t+1}\right)|x_t\right) \ | \ \kappa_{t+1}, \bar{\kappa}_{t+1} \in I^{\kappa}\} \\
&\qquad = \left\{\left(\frac{\beta\rho}{1-\beta\rho}\bar{\kappa}_{t+1} + \kappa_{t+1}\right)\rho x_t \,\middle|\, \kappa_{t+1}, \bar{\kappa}_{t+1} \in I^{\kappa}\right\} \\
&\qquad = \begin{cases} \left[\dfrac{\kappa_L\rho}{1-\beta\rho}x_t, \dfrac{\kappa_U\rho}{1-\beta\rho}x_t\right] & \text{if } x_t \geq 0, \\ \left[\dfrac{\kappa_U\rho}{1-\beta\rho}x_t, \dfrac{\kappa_L\rho}{1-\beta\rho}x_t\right] & \text{if } x_t < 0. \end{cases}
\end{aligned} \tag{8}$$

This set of conditional expectations formalizes the Knightian uncertainty about π_{t+1}.

The representation of participants' inflation expectations, $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_t)|x_t)$ in (6), is con-

sistent with the model's predictions, in the sense that

$$\mathrm{E}\left(\pi_{t+1}\left(\bar{\kappa}_t\right)|x_t\right) \in \left\{\mathrm{E}\left(\pi_{t+1}\left(\kappa_{t+1}, \bar{\kappa}_{t+1}\right)|x_t\right) \mid \kappa_{t+1}, \bar{\kappa}_{t+1} \in I^{\kappa}\right\}, \tag{9}$$

for any value of the subjective parameter $\bar{\kappa}_t \in I^{\kappa}$.

The proof of the proposition is presented in Appendix A.

Importantly, the representation of participants' inflation expectations, $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_t|x_t) = \rho\bar{\kappa}_t(1-\beta\rho)x_t$, is consistent with the model's prediction of future inflation for all values of the subjective parameter $\bar{\kappa}_t$ within the interval I^{κ}. Thus, KMH recognizes that there are many model-consistent, and thus rational, ways for participants to form expectations of future inflation. In Section 3, we show that the model-consistent representation of the market's aggregate inflation expectations, $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_t|x_t)$, is compatible with the apparent diversity of participants' expectations, even when all participants base their expectations on full information, and that the subjective parameter $\bar{\kappa}_t$ can be interpreted as representing the average of participants' subjective assessments.

2.2.4 Knightian Uncertainty as Unresolvable Ambiguity

The set of conditional expectations in (8) formalizes the Knightian uncertainty about future inflation: there is no way to reduce this uncertainty ex ante to a single probability distribution, because it arises from unforeseeable change.

Nowadays, the terms Knightian uncertainty and ambiguity are often used interchangeably to refer to models that formalize the uncertainty about future outcomes with a set of probability distributions. However, this ignores Knight's (1921) original argument that unforeseeable structural change implies that this uncertainty cannot be reduced to a single distribution ex ante. KMH formalizes this idea. Consequently, its formalization of Knightian uncertainty about future inflation, as defined by the set of conditional expectations in (8), corresponds explicitly to the specific type of ambiguity that Epstein and Schneider (2007, p. 2726) characterize as "not vanish[ing] in the long run, since the time-varying features remain impossible to know even after many observations."[7]

Epstein and Schneider contrast such unresolvable ambiguity with ambiguity that can be resolved by analyzing past data. This distinction is crucial for representing market participants' expectations: if the ambiguity could be resolved by analyzing past time-series data, rational participants would do so. Thus, to acknowledge participants' rationality in a model with resolvable ambiguity, an economist would have to represent their expectations as evolving according to a mechanism that would resolve the ambiguity asymptotically, such as Bayesian learning. In contrast, because KMH formalizes Knightian uncertainty as arising from unforeseeable change, it recognizes that rational participants can never reduce the ambiguity about future inflation to a single conditional expectation.

[7]See Ilut and Schneider (2022) for a recent survey of ambiguity in macroeconomics and finance.

2.3 The Influence of Psychological Factors on Rational Participants' Inflation Expectations

By opening the model to unforeseeable change, KMH reconciles a model-consistent representation of participants' inflation expectations with Keynes' (1936, chapter 12) argument that when the future is inherently unknowable, psychological factors play a crucial role in rational participants' expectation formation and decision-making.

When facing Knightian uncertainty, rational participants might rely on various formal and informal methods to assess future changes and form expectations of future inflation. These might include statistical estimates of past and current objective parameters, which, however, can be imprecise in the periods following shifts in κ_t, given the difficulty of identifying and estimating these shifts in real time. But even with a perfect or imperfect estimate of the current objective parameter, κ_t, rational participants must assess whether the parameter will change in the future, and if so, by what magnitude. Because the change is unforeseeable, it cannot be objectively assessed ex ante. Thus, in assessing future changes and forming their expectations of future inflation, rational participants might also rely on psychological and other factors outside the model in (1) and (2), such as market sentiment, narratives (Shiller, 2019, and Mangee, 2021), or selective recall (Tversky and Kahneman, 1973, Kahana, 2012, and Bordalo et al., 2021) of historical episodes that mimic current inflation developments.

As Keynes (1936, p. 163-164) put it: "We are merely reminding ourselves that human decisions affecting the future (...) cannot depend on strict mathematical expectation, since the basis for making such calculations does not exist; (...) our rational selves choosing between the alternatives as best we are able, calculating where we can, but often falling back for our motive on whim or sentiment or chance."

Behavioral economists have presented important experimental and empirical evidence that market participants' expectations are indeed influenced by psychological factors. Behavioral models formalize these influences with a constant mechanism that distorts participants' expectations relative to REH. For example, Gennaioli and Shleifer's (2018) diagnostic expectations formalize the influence of Kahneman and Tversky's (1972) representativeness heuristic with a constant rule that implies that participants always overreact (or underreact, depending on a parameter value) to news relative to full-information REH. Because diagnostic expectations represents participants' expectations as inconsistent with the model's predictions, it presumes that participants are irrational.

In contrast, KMH does not presume that the influence of psychological factors on participants' inflation expectations can be formalized with a constant rule. Instead, we assume that psychological factors influence participants' subjective parameters to lie within a subinterval of the objective parameter's interval. Psychological factors thus do not completely determine which model-consistent subjective expectations represent participants' inflation expectations; they only restrict this representation to a subset of the subjective expectations that are consistent with the model's predictions.

As a simple illustration, we could assume that participants' subjective parameter $\bar{\kappa}_t$ lies within a subinterval of I^κ, depending on market sentiment, as measured, for example, by the University of Michigan's Consumer Sentiment Index, which we denote by s_t. Specifically, assume that during periods when sentiment is pessimistic, $s_t < \delta$ for some threshold value δ, participants expect higher future inflation, as they assess the future pass-through to exceed its current value, and that they expect lower future inflation when sentiment is optimistic. This can be formalized by assuming that $\bar{\kappa}^{(j)} \in (\kappa^{(i)}, \kappa_U]$ for some $i, j > 0$ when $s_t < \delta$, such that $\bar{\kappa}_t > \kappa_t$ during those periods, and $\bar{\kappa}^{(j)} \in [\kappa_L, \kappa^{(i)})$ for some $i, j > 0$ when $s_t > \delta$, such that $\bar{\kappa}_t < \kappa^{(i)}$ during those periods.

2.4 The Autonomous Role of Participants' Expectations in Driving Inflation

Acknowledging that participants face Knightian uncertainty enables KMH to reconcile a model-consistent representation of rational participants' inflation expectations with the conjecture that motivated Phelps' (1970) groundbreaking micro-foundations volume: understanding how inflation evolves requires according rational participants' expectations an autonomous role relative to the economist's specification of the inflation process.

KMH formalizes the idea that participants "construct expectations of the state of the economy – over space and over time – and maximize relative to that imagined world" (Phelps, 1970, p. 22). It does so by representing rational participants' inflation expectations based on their subjective assessment of how future inflation will unfold. This is formalized in terms of the subjective parameters that are exogenous and constrained only to lie within the objective parameters' interval. Thus, KMH accords rational participants' inflation expectations an autonomous role in driving inflation, relative to the model's specification of the inflation process. Although the specification of the inflation process in (1) and (2) determines the functional form and relevant information of participants' model-consistent expectations, the level and shifts in the subjective parameters occur autonomously of the level and shifts in the objective parameters.

This implies that the specification of the inflation process in (1) and (2) does not completely determine how new information affects inflation. In the baseline NKPC model we consider here, new information arises solely in terms of shocks to the output gap, ϵ_t. These shocks feed into inflation with a direct pass-through determined by the objective parameter, κ_t. However, the shocks also feed into inflation through participants' inflation expectations. The size of this effect is autonomous, relative to the specification of the inflation process, because the revision of participants' inflation expectations in response to news depends on participants' subjective parameter, $\bar{\kappa}_t$.

During periods when participants subjective assessment of the future pass-through exceeds its current level, $\bar{\kappa}_t > \kappa_t$, the direct effect on inflation of the shock to the output gap will be amplified by the autonomous revision of participants' inflation expectations. Thus, during these periods, rational participants "overreact" to the news. In contrast, rational participants "underreact" to news about the output gap during periods where their subjective assessment of the future pass-through is lower than its current level, $\bar{\kappa}_t < \kappa_t$. Importantly, because both the objective and subjective parameters undergo nonrepetitive shifts over time, KMH recognizes that an economist cannot know in advance whether participants will over- or underreact to new information about the output gap in the future.

In contrast to KMH, participants' inflation expectations do not play an autonomous role in driving inflation in REH models, as expectations are fully determined by the model's specification of the inflation and output-gap processes.[8] As Thomas Sargent has put it, "[i]n rational expectations models, people's beliefs are among the outcomes. They are not inputs" (Evans and Honkapohja, 2005, p. 566). This implies that the specification of the inflation process completely determines how new information—here shocks to the output gap—affects inflation, and that the revision of participants' inflation expectations is always proportional to the direct pass-through of the output gap to inflation.

In models that assume no or foreseeable change, according an autonomous role to participants' inflation expectations in driving inflation requires representing these expectations as inconsistent with the model's predictions, thereby implying that participants are irrational. Indeed, because the models presented in the Phelps (1970) volume assumed constant parameters, they had to rely on model-inconsistent representations, such as adaptive expectations, to accord participants' expectations an autonomous role in driving inflation.

2.5 Rational Participants' Nonrepetitive Systematic Forecast Errors

Because KMH acknowledges that rational participants face Knightian uncertainty and thus cannot have perfect probabilistic foresight, it represents rational participants' inflation expectations as generally deviating from the ex post reduced-form inflation outcome. As a result, KMH predicts that rational market participants' ex post forecast errors are, in general, biased and correlated with ex ante information. Importantly, however, KMH also predicts that this bias and correlation will shift over time as the inflation process and participants' inflation expectations undergo nonrepetitive, and thus unforeseeable, change.

[8] For an extensive discussion of the difficulty of reconciling participants' rationality with an autonomous role for their expectations in driving inflation in models that assume that change in inflation is foreseeable, see Frydman and Phelps (2013).

For a one-period forecast horizon, the KMH model implies that participants' ex post inflation forecast errors, defined as the deviation between ex post inflation and ex ante inflation expectations, are given by:

$$\begin{aligned}\pi_{t+1}(\kappa_{t+1},\bar{\kappa}_t) - \mathrm{E}\left(\pi_{t+1}(\bar{\kappa}_t)|x_t\right) &= \left(\frac{\beta\rho}{1-\beta\rho}\bar{\kappa}_{t+1} + \kappa_{t+1}\right)x_{t+1} - \frac{\rho\bar{\kappa}_t}{1-\beta\rho}x_t \\ &= \frac{(\kappa_{t+1}-\bar{\kappa}_t) + \beta\rho(\bar{\kappa}_{t+1}-\kappa_{t+1})}{1-\beta\rho}\rho x_t + u_{t+1}, \qquad (10)\end{aligned}$$

where $u_{t+1} = ((\beta\rho\bar{\kappa}_{t+1})(1-\beta\rho)^{-1} + \kappa_{t+1})\epsilon_{t+1}$ has mean zero and is uncorrelated with all information available at time t.

During a period where $\kappa_{t+1} = \kappa^{(i)}$ and $\bar{\kappa}_t = \bar{\kappa}_{t+1} = \bar{\kappa}^{(j)}$ for some $i, j > 0$, the ex post forecast error reduces to:

$$\pi_{t+1}(\kappa^{(i)},\bar{\kappa}^{(j)}) - \mathrm{E}\left(\pi_{t+1}(\bar{\kappa}^{(j)})|x_t\right) = (\kappa^{(i)} - \bar{\kappa}^{(j)})\rho x_t + u_{t+1}, \qquad (11)$$

where $u_{t+1} = ((\beta\rho\bar{\kappa}^{(j)})(1-\beta\rho)^{-1} + \kappa^{(i)})\epsilon_{t+1}$.

This shows that participants' ex post forecast errors are generally correlated with the ex ante output gap, x_t, but that this correlation shifts in unforeseeable ways over time. During each subperiod, the size and sign of the correlation depend on the difference between the subjective parameters at times t and $t+1$ and the objective parameter at time $t+1$. For example, during a subperiod in which the subjective and objective parameters remain constant, $\bar{\kappa}_t = \bar{\kappa}_{t+1} = \bar{\kappa}^{(j)}$ and $\kappa_{t+1} = \kappa^{(i)}$ for some $j > 0$ and $i > 0$, the correlation between ex post inflation and the ex ante output gap converges asymptotically towards $(\kappa^{(i)} - \bar{\kappa}^{(j)})\rho$. This correlation is negative if participants' assessment of the future pass-through exceeds its current level, $\bar{\kappa}^{(j)} > \kappa^{(i)}$, positive if their assessment is smaller, $\bar{\kappa}^{(j)} < \kappa^{(i)}$, and zero if their assessment corresponds to its current level, $\bar{\kappa}^{(j)} = \kappa^{(i)}$. Thus, only during periods when participants' subjective assessment of the future pass-through parameter is identical to its current value, and it remains constant over the forecast horizon, will their ex post forecast errors be uncorrelated with current information.

In Frydman and Tabor (2024), we formally show, using a simple KMH model, that the parameters of the regressions of participants' ex post inflation forecast errors on either their ex ante inflation forecast revisions or ex ante inflation undergo nonrepetitive structural shifts over time. We show that both the bias and the correlation of their ex post forecast errors with ex ante information shift between values that can be positive, negative, and zero during different periods depending on the relationship between the model's objective and subjective parameters.

Coibion and Gorodnichenko (2015) presented empirical findings that survey-based ex post inflation forecast errors are positively correlated with ex ante inflation forecast revisions for the sample period from 1969 to 2014. These findings are inconsistent with constant-parameter FIRE's prediction that the forecast errors should be unpredictable. However, Coibion and Gorodnichenko showed that they are consistent with the predictions of constant-parameter REH models with sticky- or noisy information, as formalized by Mankiw and Reis (2002), Woodford (2003), Sims (2003), and Mackowiak and Wiederholt (2009). This led them to conclude that the empirical rejection of FIRE is "unlikely to be driven by departures from rationality (...) and instead reflect deviations from the assumption of full-information" (Coibion and Gorodnichenko, 2015, p. 2646).

However, Angeletos, Huo, and Sastry (2021), Hajdini and Kurmann (2024), and Frydman and Tabor (2024) showed that the estimated correlation of ex post inflation forecast errors with ex ante forecast revisions becomes insignificantly different from zero when the sample period is changed to start in the early 1980s. Moreover, the estimated correlation is significantly positive for the 1970s subsample, which suggests that Coibion and Gorodnichenko's positive estimate for the full sample from 1969 to 2014 is driven by the observations with high ex post forecast errors and ex ante forecast revisions during the 1970s, and not a general feature characterizing participants' survey-based inflation expectations.

Hajdini and Kurmann (2024) show that in a model with Markov switching between two repetitive regimes, FIRE predicts that the bias and correlation of participants' ex post inflation forecast errors with ex ante information should switch between repetitive positive and negative regimes. They show that these underlying regime switches can give rise to Coibion and Gorodnichenko's (2015) empirical finding of a positive correlation between ex post inflation forecast errors and ex ante forecast revisions when the correlation is estimated as constant. Moreover, they estimate a DSGE model with Markov switching between two repetitive regimes and inflation expectations represented by FIRE, simulate time-series data from the estimated model, and estimate the forecast-error regressions for the simulated data. Although the estimates based on the simulated data vary over time, they conclude that these variations "are generally quite different from the ones we observe in the (Survey of Professional Forecasters) data" (Hajdini and Kurmann, 2024, p. 35).

In Frydman and Phelps (2013), we used step-indicator and multiplicative step-indicator saturation with the Autometrics model-selection algorithm (see Doornik, 2009, Castle, Doornik, and Hendry, 2012, and Castle et al., 2015) to identify the timing of structural shifts in inflation forecast-error regressions from the time-series data. Our empirical findings provide support for KMH's prediction that the bias and correlation of rational participants' ex post inflation forecast errors with ex ante information should shift between nonrepetitive values. Indeed, we found multiple structural shifts in both the estimate of the bias and the correlations. Most of these shifts and the largest biases and correlations occurred during the 1970s and after the COVID-19 pandemic in 2020. These were periods characterized by major structural changes in monetary policy and inflation dynamics. In contrast, we estimated the bias to be small and the correlation with ex ante information to be insignificantly different from zero during most of the Great Moderation period from the early 1980s until 2020. This is widely considered a stable period without major structural changes in monetary policy and inflation dynamics. Consistent with KMH's predictions, these findings indicate that the bias and correlation of survey-based inflation forecast errors with ex ante information arise from unforeseeable change and the Knightian uncertainty about inflation arising from such change.

3 Reconciling Rationality with the Diversity of Participants' Inflation Expectations

In this section, we show how opening a macroeconomic model to unforeseeable change enables KMH to reconcile a model-consistent representation of rational participants' inflation expectations with the apparent diversity of their expectations, even when all participants have access to and base their expectations on full information.

To this end, we modify the NKPC model in the previous section so that inflation depends on the average of N participants' inflation expectations. We assume all participants have access to full information but represent their model-consistent expectations in terms of different subjective parameters. Thus, KMH recognizes that rational market participants who face Knightian uncertainty arising from unforeseeable change form diverse expectations because they base their expectations on different subjective understandings of how future inflation will unfold.

We also show that the average of all N participants' diverse model-consistent inflation expectations is identical to the model-consistent representation of the market's aggregate inflation expectations in the KMH model presented in Section 2. Thus, the market's aggregate subjective parameter, $\bar{\kappa}_t$ in (4) can be interpreted as the average of all N participants' diverse subjective parameters. The KMH model thus provides a simple, tractable way to model inflation and the market's aggregate inflation expectations that is compatible with the underlying diversity of individual participants' inflation expectations.

3.1 A Baseline NKPC Model with Diverse Expectations

We modify the baseline NKPC in (1) to depend on the average of N market participants' inflation expectations, so that inflation is given by:

$$\pi_t = \beta \frac{1}{N} \sum_{n=1}^{N} \mathrm{E}\left(\pi_{t+1}\left(\bar{\kappa}_{n,t}\right)|x_t\right) + \kappa_t x_t, \tag{12}$$

for $t = 1, 2, \ldots$, and where $0 < \beta < 1$, the output gap x_t evolves according to the constant-parameter stationary autoregressive process in (2), κ_t undergoes nonrepetitive, and thus unforeseeable, change as formalized in (3), and $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_{n,t})|x_t)$ denotes participant n's model-consistent inflation expectation represented in terms of the subjective parameter $\bar{\kappa}_{n,t}$ for $n = 1, 2, \ldots, N$.

Over time, we assume that participant n's subjective parameter intermittently shifts between nonrepetitive, and thus unforeseeable, regimes restricted to lie within the objective parameter's interval, I^κ. We formalize this by:

$$\bar{\kappa}_{n,t} = \bar{\kappa}^{(n,j)} \in I^\kappa = [\kappa_L, \kappa_U], \quad t \in [\bar{\tau}_{j-1}, \bar{\tau}_j - 1], \quad n = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, \tag{13}$$

where $\bar{\tau}_j > \bar{\tau}_{j-1}$ for all j and $\bar{\tau}_0 = 1$. Here, we formalize the nonrepetitive shifts to occur at the same times for all participants, but we can allow them to occur at different times without a loss of generality by allowing $\bar{\kappa}^{(n,j)} = \bar{\kappa}^{(n,j-1)}$ for some j.

KMH hypothesizes that participant n's inflation expectation can be represented with a subjective conditional expectation. We define this as the conditional expectation of the next period's inflation implied by (12) and (2), but where the future objective parameter, κ_{t+h} for $h = 1, 2, \ldots$, and all participants' future subjective parameters, $\bar{\kappa}_{m,t+h}$ for $m = 1, 2, \ldots, N$ and $h = 1, 2, \ldots$, are replaced by the participant's subjective parameter, $\bar{\kappa}_{n,t}$. Let $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_{n,t}|x_t)$ denote this conditional expectation, where $\pi_{t+h}(\bar{\kappa}_{n,t})$ is given by the subjective process:

$$\pi_{t+h}\left(\bar{\kappa}_{n,t}\right) = \beta \frac{1}{N} \sum_{m=1}^{N} \mathrm{E}\left(\pi_{t+h+1}\left(\bar{\kappa}_{n,t}\right)|x_{t+h}\right) + \bar{\kappa}_{n,t} x_{t+h} \tag{14}$$

for fixed t, $h = 1, 2, \ldots$, and $n = 1, 2, \ldots, N$.

Here, the subjective parameter, $\bar{\kappa}_{n,t}$, represents each participant's subjective assessment of how the output gap will pass through to inflation next period. Importantly, it also represents the participant's subjective assessment at time t of the average of all participants' subjective assessments at $t+1$ of κ_{t+2}, and similar higher-order subjective assessments. Here, we have made the simplifying assumption that these subjective assessments are identical.[9]

The following proposition states the expression for the model-consistent representation of participant n's subjective inflation expectation $E(\pi_{t+1}(\bar{\kappa}_{n,t})|x_t)$ in (12).

Proposition 3. *Let* $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_{n,t})|x_t)$ *denote the model-consistent subjective conditional expectation of* $\pi_{t+1}(\bar{\kappa}_{n,t})$ *for participant* $n = 1, 2, \ldots, N$, *where* $\bar{\kappa}_{n,t}$ *denotes the participant's subjective parameter at time* t, *as specified in (13), and* $\pi_{t+h}(\bar{\kappa}_{n,t})$ *denotes the participant's subjective process for future inflation, as specified in (14) for a fixed* t *and*

[9]We could, however, allow each participant's subjective assessment of the future objective parameters to differ from their higher-order subjective assessments. In that case, each participant's subjective expectation of the next period's inflation would differ from their subjective expectation of the discounted sum of how future output gaps pass through to inflation.

$h = 1, 2, \ldots$ with x_t following (2). The participant's subjective conditional expectation is uniquely given by:

$$\mathrm{E}\left(\pi_{t+1}\left(\bar{\kappa}_{n,t}\right)|x_t\right) = \frac{\rho\bar{\kappa}_{n,t}}{1-\beta\rho}x_t. \tag{15}$$

The proof of the proposition is presented in Appendix A.

The proposition shows how acknowledging that participants face Knightian uncertainty arising from unforeseeable change in κ_t enables KMH to represent participants' model-consistent inflation expectations as diverse, even when they all have access to and base their expectations on full information. In this baseline NKPC model with only a single exogenous variable and unforeseeable change only in the objective parameter κ_t, the diversity arises solely from diversity in participants' subjective parameter $\bar{\kappa}_{n,t}$. Thus, KMH recognizes that there are many rational ways for participants to translate the relevant information—here only the output gap—into subjective expectations of future inflation.

In a more general model with several variables and parameters that undergo unforeseeable change within intervals that nest zero, participants' diverse inflation expectations could be represented in terms of different variables by assuming some subjective parameters are equal to zero. That would enable KMH to represent participants' diverse inflation expectations in terms of different subjective models. These models would, however, be restricted to lie within a subset of models determined by the model's specification of the inflation process. Andre et al.'s (2022) empirical findings show that participants appear to base their expectations on different subjective models, even when they have access to the same information set.

The simultaneous specification of the NKPC in (12) and the model-consistent representation of participants' inflation expectations in (15) implies that the reduced-form expression for inflation, which we denote by $\pi_t(\kappa_t, \bar{\kappa}_t^*)$ where $\bar{\kappa}_t^* = (\bar{\kappa}_{1,t}, \bar{\kappa}_{2,t}, \ldots, \bar{\kappa}_{N,t})$, is given by:

$$\pi_t\left(\kappa_t, \bar{\kappa}_t^*\right) = \beta\left(\frac{1}{N}\sum_{n=1}^{N}\frac{\rho\bar{\kappa}_{n,t}}{1-\beta\rho}x_t\right) + \kappa_t x_t = \left(\frac{\beta\rho}{1-\beta\rho}\bar{\kappa}_{M,t} + \kappa_t\right)x_t, \tag{16}$$

where $\bar{\kappa}_{M,t} = N^{-1}\sum_{n=1}^{N}\bar{\kappa}_{n,t}$ is the average of all participants' subjective parameters.

The following proposition specifies the set of conditional expectations of the next period's reduced-form inflation that constitutes the model's predictions and formalizes the Knightian uncertainty about π_{t+1}. Moreover, the proposition shows that the representation of participant n's inflation expectations in (15) is consistent with the model's predictions, as it lies within this set for any value of $\bar{\kappa}_{n,t} \in I^{\kappa}$.

Proposition 4. *The representation of participant n's inflation expectation, $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_{n,t}))$ in (15), is consistent with the model's predictions, in the sense that it lies within the set of* *conditional expectations of the reduced-form expression $\pi_{t+1}(\kappa_{t+1}, \bar{\kappa}^*_{t+1})$, defined in (16), as given by:*

$$\left\{ E\left(\pi_{t+1}\left(\kappa_{t+1}, \bar{\kappa}^*_{t+1}\right)|x_t\right) \middle| \kappa_{t+1}, \bar{\kappa}_{n,t+1} \in I^\kappa, n = 1, 2, \ldots, N \right\}$$
$$= \left\{ \left(\frac{\beta\rho}{1-\beta\rho} \left(\frac{1}{N} \sum_{n=1}^{N} \bar{\kappa}_{n,t+1} \right) + \kappa_{t+1} \right) \rho x_t \middle| \kappa_{t+1}, \bar{\kappa}_{n,t+1} \in I^\kappa, n = 1, 2, \ldots, N \right\}$$
$$= \begin{cases} \left[\frac{\kappa_L \rho}{1-\beta\rho} x_t, \frac{\kappa_U \rho}{1-\beta\rho} x_t \right] & \textit{if } x_t \geq 0, \\ \left[\frac{\kappa_U \rho}{1-\beta\rho} x_t, \frac{\kappa_L \rho}{1-\beta\rho} x_t \right] & \textit{if } x_t < 0. \end{cases} \tag{17}$$

The proof of the proposition is presented in Appendix A.

3.2 The Compatibility of the Aggregate Market's Model-Consistent Inflation Expectations with the Diversity of Participants' Expectations

The foregoing expressions and propositions show that the KMH model with N participants holding diverse model-consistent inflation expectations closely resembles the model presented in Section 2, where the market's aggregate model-consistent inflation expectations were represented directly. Indeed, the following corollary states that the model-consistent representation of the market's aggregate inflation expectation in (6) is identical to the average of the N participants' model-consistent inflation expectations in (15).

Corollary 1. *The average of the N participants' model-consistent inflation expectations in (15),*

$$\frac{1}{N} \sum_{n=1}^{N} \mathrm{E}(\pi_{t+1}(\bar{\kappa}_{n,t})|x_t) = \frac{1}{N} \sum_{n=1}^{N} \frac{\rho\bar{\kappa}_{n,t}}{1-\beta\rho} x_t = \frac{\rho\bar{\kappa}_{M,t}}{1-\beta\rho} x_t, \tag{18}$$

where $\bar{\kappa}_{M,t} = N^{-1} \sum_{n=1}^{N} \bar{\kappa}_{n,t}$, is identical to the model-consistent representation of the market's aggregate inflation expectation in (6),

$$\mathrm{E}(\pi_{t+1}(\bar{\kappa}_t)|x_t) = \frac{\rho\bar{\kappa}_t}{1-\beta\rho} x_t, \tag{19}$$

with $\bar{\kappa}_t = \bar{\kappa}_{M,t} = N^{-1} \sum_{n=1}^{N} \bar{\kappa}_{n,t}$.

The corollary shows that although the KMH model presented in Section 2 models the market's aggregate inflation expectations, its model-consistent representation of these expectations is compatible with the underlying diversity of market participants' inflation expectations. Indeed, the market's aggregate subjective parameter, $\bar{\kappa}_t$ in (4) can be directly interpreted as representing the average of N participants' diverse subjective parameters, $\bar{\kappa}_{M,t} = N^{-1} \sum_{n=1}^{N} \bar{\kappa}_{n,t}$. Thus, although the KMH model presented in Section 2 represents the aggregate of participants' inflation expectations, it also captures the underlying diversity of participants' inflation expectations directly, through the model-consistent representation of the aggregate of participants' model-consistent inflation expectations. Thus, the model provides a simple and tractable way to model inflation when we are not interested in the diversity of expectations per se.

4 Concluding Remarks

This paper builds on the observation that the economy's structure changes in nonrepetitive ways that no one can objectively foresee, even in probabilistic terms. Here, we have focused on structural change in inflation.

There is ample empirical evidence of such change in inflation. Surveying the literature that explores changes in the New Keynesian Phillips curve over time, Hooper, Mishkin, and Sufi (2020, p. 28) conclude that "[t]here is a strong consensus in the literature that US inflation dynamics have changed dramatically over the past several decades." Recently, Smith, Timmermann, and Wright (2023, and references therein) found structural shifts in the Phillips curve for the United States and the European Union arising from a "complex set of factors" (p. 2) that are, at least in part, nonrepetitive. Ball, Leigh, and Mishra (2022) and others have found that the structure of the inflation process changed substantially at the onset of the COVID-19 pandemic, which was clearly an event whose consequences for inflation could not have been foreseen. Finally, Hendry and Doornik (2014), Hendry (2018), and other studies they cite have demonstrated empirically that time series of macroeconomic outcomes undergo structural shifts at times and with magnitudes that are nonrepetitive.

KMH is a general approach that can be used to formalize unforeseeable change in any macroeconomic model. Here we have shown how unforeseeable change in a baseline New Keynesian Phillips curve model's pass-through of the output gap to inflation implies unforeseeable change in rational participants' inflation expectations and inflation. Whether implementing KMH in a more general model of inflation can account for the foregoing empirical findings and deriving its implications for policy analysis is an avenue for future research.

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A Proof of Propositions

Proof of Proposition 1. We derive the expression for the model-consistent subjective expectation, $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_t)|x_t)$, by forward iteration with respect to the subjective process for future inflation, $\pi_{t+h}(\bar{\kappa}_t)$ for t fixed and $h = 1, 2, \ldots$, as given by (5). To do so, we rely on the law of iterated expectations, which holds for that process. We also use $0 < \beta < 1$ and that the process for x_t in (2), with $0 < \rho < 1$, implies that $\mathrm{E}(x_{t+h}|x_t) = \rho^h x_t$ for $h > 0$. The forward iteration yields:

$$\begin{aligned}
\mathrm{E}\left(\pi_{t+1}\left(\bar{\kappa}_t\right)|x_t\right) &= \mathrm{E}\left(\beta\,\mathrm{E}\left(\pi_{t+2}\left(\bar{\kappa}_t\right)|x_{t+1}\right) + \bar{\kappa}_t x_{t+1}|x_t\right)\\
&= \beta\,\mathrm{E}\left(\pi_{t+2}\left(\bar{\kappa}_t\right)|x_t\right) + \bar{\kappa}_t\,\mathrm{E}\left(x_{t+1}|x_t\right)\\
&= \beta\,\mathrm{E}\left(\beta\,\mathrm{E}\left(\pi_{t+3}\left(\bar{\kappa}_t\right)|x_{t+2}\right) + \bar{\kappa}_t x_{t+2}|x_t\right) + \bar{\kappa}_t\,\mathrm{E}\left(x_{t+1}|x_t\right)\\
&= \beta^2\,\mathrm{E}\left(\pi_{t+3}\left(\bar{\kappa}_t\right)|x_t\right) + \beta\bar{\kappa}_t\,\mathrm{E}\left(x_{t+2}|x_t\right) + \bar{\kappa}_t\,\mathrm{E}\left(x_{t+1}|x_t\right)\\
&\vdots\\
&= \beta^K\,\mathrm{E}\left(\pi_{t+K+1}\left(\bar{\kappa}_t\right)|x_t\right) + \sum_{h=1}^{K}\beta^{h-1}\bar{\kappa}_t\,\mathrm{E}\left(x_{t+h}|x_t\right)\\
&= \beta^K\,\mathrm{E}\left(\pi_{t+K+1}\left(\bar{\kappa}_t\right)|x_t\right) + \sum_{h=1}^{K}\beta^{h-1}\bar{\kappa}_t\rho^h x_t\\
&= \beta^K\,\mathrm{E}\left(\pi_{t+K+1}\left(\bar{\kappa}_t\right)|x_t\right) + \sum_{h=0}^{K-1}(\beta\rho)^h\rho\bar{\kappa}_t x_t. \qquad (20)
\end{aligned}$$

Since the assumptions $0 < \beta < 1$ and $0 < \rho < 1$ imply that $0 < \beta\rho < 1$, it follows that $\lim_{K\to\infty}\sum_{h=0}^{K-1}(\beta\rho)^h = (1-\beta\rho)^{-1}$ and

$$\lim_{K\to\infty}\beta^K\,\mathrm{E}(\pi_{t+K+1}(\bar{\kappa}_t)|x_t) = 0, \qquad (21)$$

so that the expression for $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_t)|x_t)$ in (20) converges, for $K \to \infty$, to:

$$\mathrm{E}\left(\pi_{t+1}\left(\bar{\kappa}_t\right)|x_t\right) = \frac{\rho\bar{\kappa}_t}{1-\beta\rho}x_t. \qquad (22)$$

■

Proof of Proposition 2. Given the objective and subjective parameters κ_{t+1} and $\bar{\kappa}_{t+1}$, the following is the reduced-form expression for $\pi_{t+1}(\kappa_{t+1}, \bar{\kappa}_{t+1})$:

$$\pi_{t+1}\left(\kappa_{t+1}, \bar{\kappa}_{t+1}\right) = \left(\frac{\beta\rho\bar{\kappa}_{t+1}}{1-\beta\rho} + \kappa_{t+1}\right)x_{t+1}. \qquad (23)$$

This implies that the conditional expectation $\mathrm{E}(\pi_{t+1}(\kappa_{t+1},\bar{\kappa}_{t+1})|x_t)$ is given by:

$$\begin{aligned}
\mathrm{E}\left(\pi_{t+1}\left(\kappa_{t+1},\bar{\kappa}_{t+1}\right)|x_t\right) &= \mathrm{E}\left(\left(\frac{\beta\rho\bar{\kappa}_{t+1}}{1-\beta\rho}+\kappa_{t+1}\right)x_{t+1}\middle|x_t\right) \\
&= \left(\frac{\beta\rho\bar{\kappa}_{t+1}}{1-\beta\rho}+\kappa_{t+1}\right)\mathrm{E}\left(x_{t+1}|x_t\right) \\
&= \left(\frac{\beta\rho\bar{\kappa}_{t+1}}{1-\beta\rho}+\kappa_{t+1}\right)\rho x_t. \qquad (24)
\end{aligned}$$

Because $0<\beta<1$ and $0<\rho<1$, $\beta\rho(1-\beta\rho)^{-1}>0$, this implies that $(\frac{\beta\bar{\kappa}_{t+1}}{1-\beta\rho}+\kappa_{t+1})$ is increasing in both κ_{t+1} and $\bar{\kappa}_{t+1}$. Moreover, as $0<\kappa_L\le\kappa_{t+1}\le\kappa_U$ and $0<\kappa_L\le\bar{\kappa}_{t+1}\le\kappa_U$, it follows that

$$\frac{\beta\rho\bar{\kappa}_{t+1}}{1-\beta\rho}+\kappa_{t+1}\ge\frac{\beta\rho\kappa_L}{1-\beta\rho}+\kappa_L=\frac{\kappa_L}{1-\beta\rho},$$

and

$$\frac{\beta\rho\bar{\kappa}_{t+1}}{1-\beta\rho}+\kappa_{t+1}\le\frac{\beta\rho\kappa_U}{1-\beta\rho}+\kappa_U=\frac{\kappa_U}{1-\beta\rho}.$$

Thus, it follows that the set of conditional expectations of $\pi_{t+1}(\kappa_{t+1},\bar{\kappa}_{t+1})$ for all possible values of $\kappa_{t+1},\bar{\kappa}_{t+1}\in I^\kappa=[\kappa_L,\kappa_U]$ is given by:

$$\begin{aligned}
&\left\{E\left(\pi_{t+1}\left(\kappa_{t+1},\bar{\kappa}_{t+1}\right)|x_t\right)\;\middle|\;\kappa_{t+1},\bar{\kappa}_{t+1}\in I^\kappa\right\} \\
&\quad=\left\{\left(\frac{\beta\rho}{1-\beta\rho}\bar{\kappa}_{t+1}+\kappa_{t+1}\right)\rho x_t\;\middle|\;\kappa_{t+1},\bar{\kappa}_{t+1}\in I^\kappa\right\} \\
&\quad=\begin{cases}\left[\dfrac{\kappa_L\rho}{1-\beta\rho}x_t,\dfrac{\kappa_U\rho}{1-\beta\rho}x_t\right] & \text{if } x_t\ge 0,\\[2ex] \left[\dfrac{\kappa_U\rho}{1-\beta\rho}x_t,\dfrac{\kappa_L\rho}{1-\beta\rho}x_t\right] & \text{if } x_t<0.\end{cases} \qquad (25)
\end{aligned}$$

Moreover, it follows from $\bar{\kappa}_t\in I^\kappa=[\kappa_L,\kappa_U]$ that participants' subjective conditional expectation, $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_t)|x_t)=\rho\bar{\kappa}_t(1-\beta\rho)^{-1}x_t$ in (6), lies within the set of conditional expectations of $\pi_{t+1}(\kappa_{t+1},\bar{\kappa}_t)$ in (25). ■

Proof of Proposition 3. We derive the expression for the subjective expectation of participant n, $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_{n,t})|x_t)$ for $n=1,2,\ldots,N$, by forward iteration with respect to the participant's subjective process for future inflation, $\pi_{t+h}(\bar{\kappa}_{n,t})$ for fixed t and $h=1,2,\ldots$, as given specified in (14). To do so, we rely on the law of iterated expectations, which holds for that process. We use $0<\beta<1$ and the process for x_t in (2) with $0<\rho<1$, which implies that $\mathrm{E}(x_{t+h}|x_t) = \rho^h x_t$ for $h > 0$. The forward iteration yields:

$$\begin{aligned}
\mathrm{E}\left(\pi_{t+1}\left(\bar{\kappa}_{n,t}\right)|x_t\right) &= \mathrm{E}\left(\beta \frac{1}{N}\sum_{m=1}^{N}\mathrm{E}\left(\pi_{t+2}\left(\bar{\kappa}_{n,t}\right)|x_{t+1}\right) + \bar{\kappa}_{n,t}x_{t+1}\middle| x_t\right) \\
&= \mathrm{E}\left(\beta \frac{1}{N}\sum_{m=1}^{N}\mathrm{E}\left(\pi_{t+2}\left(\bar{\kappa}_{n,t}\right)|x_{t+1}\right)\middle| x_t\right) + \bar{\kappa}_{n,t}\,\mathrm{E}\left(x_{t+1}|x_t\right) \\
&= \beta\,\mathrm{E}\left(\mathrm{E}\left(\pi_{t+2}\left(\bar{\kappa}_{n,t}\right)|x_{t+1}\right)|x_t\right) + \bar{\kappa}_{n,t}\,\mathrm{E}\left(x_{t+1}|x_t\right) \\
&= \beta\,\mathrm{E}\left(\pi_{t+2}\left(\bar{\kappa}_{n,t}\right)|x_t\right) + \bar{\kappa}_{n,t}\,\mathrm{E}\left(x_{t+1}|x_t\right) \\
&= \beta\,\mathrm{E}\left(\beta \frac{1}{N}\sum_{m=1}^{N}\mathrm{E}\left(\pi_{t+3}\left(\bar{\kappa}_{n,t}\right)|x_{t+2}\right) + \bar{\kappa}_{n,t}x_{t+2}\middle| x_t\right) + \bar{\kappa}_{n,t}\,\mathrm{E}\left(x_{t+1}|x_t\right) \\
&= \beta\,\mathrm{E}\left(\beta \frac{1}{N}\sum_{m=1}^{N}\mathrm{E}\left(\pi_{t+3}\left(\bar{\kappa}_{n,t}\right)|x_{t+2}\right)\middle| x_t\right) + \beta\bar{\kappa}_{n,t}\,\mathrm{E}\left(x_{t+2}|x_t\right) + \bar{\kappa}_{n,t}\,\mathrm{E}\left(x_{t+1}|x_t\right) \\
&= \beta^2\,\mathrm{E}\left(\pi_{t+3}\left(\bar{\kappa}_{n,t}\right)|x_t\right) + \beta\bar{\kappa}_{n,t}\,\mathrm{E}\left(x_{t+2}|x_t\right) + \bar{\kappa}_{n,t}\,\mathrm{E}\left(x_{t+1}|x_t\right) \\
&\vdots \\
&= \beta^K\,\mathrm{E}\left(\pi_{t+K+1}\left(\bar{\kappa}_{n,t}\right)|x_t\right) + \sum_{h=1}^{K}\beta^{h-1}\bar{\kappa}_{n,t}\,\mathrm{E}\left(x_{t+h}|x_t\right) \\
&= \beta^K\,\mathrm{E}\left(\pi_{t+K+1}\left(\bar{\kappa}_{n,t}\right)|x_t\right) + \sum_{h=1}^{K}\beta^{h-1}\bar{\kappa}_{n,t}\rho^h x_t \\
&= \beta^K\,\mathrm{E}\left(\pi_{t+K+1}\left(\bar{\kappa}_{n,t}\right)|x_t\right) + \sum_{h=0}^{K-1}(\beta\rho)^h\rho\bar{\kappa}_{n,t}x_t. \qquad (26)
\end{aligned}$$

Since the assumptions $0 < \beta < 1$ and $0 < \rho < 1$ imply that $0 < \beta\rho < 1$, it follows that $\lim_{K\to\infty}\sum_{h=0}^{K-1}(\beta\rho)^h = (1-\beta\rho)^{-1}$ and

$$\lim_{K\to\infty}\beta^K\,\mathrm{E}(\pi_{t+K+1}(\bar{\kappa}_{n,t})|x_t) = 0, \qquad (27)$$

so that the expression for $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_{n,t})|x_t)$ in (26) converges, for $K\to\infty$, to:

$$\mathrm{E}\left(\pi_{t+1}\left(\bar{\kappa}_{n,t}\right)|x_t\right) = \frac{\rho\bar{\kappa}_{n,t}}{1-\beta\rho}x_t. \qquad (28)$$

■

Proof of Proposition 4. Given the objective and subjective parameters κ_{t+1} and $\bar{\kappa}^*_{t+1} = (\bar{\kappa}_{1,t+1}, \bar{\kappa}_{2,t+1}, \ldots, \bar{\kappa}_{N,t+1})$, the following is the reduced-form expression for $\pi_{t+1}(\kappa_{t+1}, \bar{\kappa}^*_{t+1})$:

$$\pi_{t+1}\left(\kappa_{t+1}, \bar{\kappa}^*_{t+1}\right) = \left(\frac{\beta\rho}{1-\beta\rho}\left(N^{-1}\sum_{n=1}^{N}\bar{\kappa}_{n,t+1}\right) + \kappa_{t+1}\right)x_{t+1}. \qquad (29)$$

This implies that the conditional expectation $\mathrm{E}(\pi_{t+1}(\kappa_{t+1}, \bar{\kappa}^*_{t+1})|x_t)$ is given by:

$$\begin{aligned}
\mathrm{E}\left(\pi_{t+1}\left(\kappa_{t+1}, \bar{\kappa}^*_{t+1}\right)|x_t\right) &= \mathrm{E}\left(\left(\frac{\beta\rho}{1-\beta\rho}\left(N^{-1}\sum_{n=1}^{N}\bar{\kappa}_{n,t+1}\right)+\kappa_{t+1}\right)x_{t+1}\middle|x_t\right)\\
&= \left(\frac{\beta\rho}{1-\beta\rho}\left(N^{-1}\sum_{n=1}^{N}\bar{\kappa}_{n,t+1}\right)+\kappa_{t+1}\right)\mathrm{E}\left(x_{t+1}|x_t\right)\\
&= \left(\frac{\beta\rho}{1-\beta\rho}\left(N^{-1}\sum_{n=1}^{N}\bar{\kappa}_{n,t+1}\right)+\kappa_{t+1}\right)\rho x_t. \qquad (30)
\end{aligned}$$

Because $0 < \beta < 1$ and $0 < \rho < 1$, so that $\beta\rho(1-\beta\rho)^{-1} > 0$, this implies that $\mathrm{E}(\pi_{t+1}(\kappa_{t+1}, \bar{\kappa}^*_{t+1})|x_t)$ is increasing in κ_{t+1} and $\bar{\kappa}_{n,t+1}$ for $n = 1, 2, \ldots, N$. Moreover, as $0 < \kappa_L \leq \kappa_{t+1} \leq \kappa_U$ and $0 < \kappa_L \leq \bar{\kappa}_{n,t+1} \leq \kappa_U$ for all $n = 1, 2, \ldots, N$, it follows that

$$\frac{\beta\rho}{1-\beta\rho}\left(N^{-1}\sum_{n=1}^{N}\bar{\kappa}_{n,t+1}\right)+\kappa_{t+1} \geq= \frac{\beta\rho\kappa_L}{1-\beta\rho}+\kappa_L = \frac{\kappa_L}{1-\beta\rho},$$

and

$$\frac{\beta\rho}{1-\beta\rho}\left(N^{-1}\sum_{n=1}^{N}\bar{\kappa}_{n,t+1}\right)+\kappa_{t+1} \leq \frac{\beta\rho\kappa_U}{1-\beta\rho}+\kappa_U = \frac{\kappa_U}{1-\beta\rho}.$$

Thus, it follows that the set of conditional expectations of $\pi_{t+1}(\kappa_{t+1}, \bar{\kappa}^*_{t+1})$ for all possible values of $\kappa_{t+1}, \bar{\kappa}_{n,t+1} \in I^\kappa = [\kappa_L, \kappa_U]$ for all $n = 1, 2, \ldots, N$ is given by:

$$\begin{aligned}
&\left\{E\left(\pi_{t+1}\left(\kappa_{t+1}, \bar{\kappa}^*_{t+1}\right)|x_t\right) \;\middle|\; \kappa_{t+1}, \bar{\kappa}_{1,t+1}, \bar{\kappa}_{2,t+1}, \ldots, \bar{\kappa}_{N,t+1} \in I^\kappa\right\}\\
&\quad= \left\{\left(\frac{\beta\rho}{1-\beta\rho}\left(N^{-1}\sum_{n=1}^{N}\bar{\kappa}_{n,t+1}\right)+\kappa_{t+1}\right)\rho x_t \;\middle|\; \kappa_{t+1}, \bar{\kappa}_{1,t+1}, \bar{\kappa}_{2,t+1}, \ldots, \bar{\kappa}_{N,t+1} \in I^\kappa\right\}\\
&\quad= \begin{cases}\left[\dfrac{\kappa_L\rho}{1-\beta\rho}x_t, \dfrac{\kappa_U\rho}{1-\beta\rho}x_t\right] & \text{if } x_t \geq 0,\\ \left[\dfrac{\kappa_U\rho}{1-\beta\rho}x_t, \dfrac{\kappa_L\rho}{1-\beta\rho}x_t\right] & \text{if } x_t < 0.\end{cases} \qquad (31)
\end{aligned}$$

Moreover, it follows from $\bar{\kappa}_{n,t} \in I^\kappa = [\kappa_L, \kappa_U]$ that each participant's subjective conditional expectation, $\mathrm{E}(\pi_{t+1}(\bar{\kappa}_{n,t})|x_t) = \rho\bar{\kappa}_{n,t}(1-\beta\rho)^{-1}x_t$ in (15) for $n = 1, 2, \ldots, N$, lies within the set of conditional expectations of $\pi_{t+1}(\kappa_{t+1}, \bar{\kappa}^*_{t+1})$ in (31). ■

B Model-Consistent Inflation Expectations with Subjective Parameters Differing Across the Forecast Horizons

In the model presented in Section 2, we assumed that participants' subjective assessment of the future objective parameters, at each time, was identical across all forecast horizons. Here, we relax this assumption and rely on KMH to derive a unique representation for participants' model-consistent subjective expectation of inflation in the general case in which participants' subjective assessment differs across all forecast horizons. Although this case is not particularly useful, as it represents participants' inflation expectations, at each point in time, in terms of an infinite sequence of subjective parameters, it nests many interesting special cases. For example, it enables an economist to assess how a shift in participants' assessment of changes in the future objective parameters affects their inflation expectations and inflation today.

We extend the model presented in Section 2 by allowing participants' subjective parameters, at time t, to shift across the forecast horizons $h = 1, 2, \ldots$. To this end, we define the vector of subjective parameters

$$\bar{\boldsymbol{\kappa}}_t = (\bar{\kappa}_{t,t+1}, \bar{\kappa}_{t,t+2}, \ldots), \qquad \bar{\kappa}_{t,t+h} \in I^{\kappa} \text{ for } h = 1, 2, \ldots, \tag{32}$$

and where $\bar{\kappa}_{t,t+h}$ shifts over time t, similarly to the specification in (4) for $h = 1, 2, \ldots$.

Here, each subjective parameter $\bar{\kappa}_{t,t+h}$ for $h = 1, 2, \ldots$ represents participants' (higher-order) assessment, at time t, of the future objective parameter κ_{t+h}. These can differ across all forecast horizons, $h > 0$, but they are all restricted to lie within the positive interval I^{κ} at all times.

Given the vector of subjective parameters, $\bar{\boldsymbol{\kappa}}_t$, we denote participants' subjective conditional expectation of the next period's inflation by $\mathrm{E}(\pi_{t+1}(\bar{\boldsymbol{\kappa}}_t)|x_t)$. We define this as the conditional expectation of π_{t+1} implied by (1) and (2), but where the future objective parameters, κ_{t+h}, are replaced by the corresponding subjective parameters, $\bar{\kappa}_{t,t+h}$ for fixed t and $h = 1, 2, \ldots$. Thus, participants' subjective process for future inflation is given by:

$$\pi_{t+h}(\bar{\boldsymbol{\kappa}}_t) = \beta\, \mathrm{E}\left(\pi_{t+h+1}(\bar{\boldsymbol{\kappa}}_t)\,|x_{t+h}\right) + \bar{\kappa}_{t,t+h} x_{t+h}, \tag{33}$$

for fixed t and $h = 1, 2, \ldots$.

We derive a unique expression for $\mathrm{E}(\pi_{t+1}(\bar{\boldsymbol{\kappa}}_t)|x_t)$ by forward iteration using the subjective process for $\pi_{t+h}(\bar{\boldsymbol{\kappa}}_t)$ for $h = 1, 2, \ldots$. in (33). Although the subjective parameters shift over the forecast horizon, their boundedness ensures that this forward iteration yields a unique expression for $\mathrm{E}(\pi_{t+1}(\bar{\boldsymbol{\kappa}}_t)|x_t)$, as stated in the following proposition.

Proposition 5. *Let* $\mathrm{E}(\pi_{t+1}(\bar{\boldsymbol{\kappa}}_t)|x_t)$ *denote participants' model-consistent subjective expec-*

tation of the next period's inflation, where $\bar{\boldsymbol{\kappa}}_t$ *denotes their vector of subjective parameters, as specified in (32), and* $\pi_{t+h}(\bar{\boldsymbol{\kappa}}_t)$ *denotes their subjective process for future inflation, as specified by (33) for fixed* t *and* $h = 1, 2, \ldots$*. Iterating forward,* $\mathrm{E}(\pi_{t+1}(\bar{\boldsymbol{\kappa}}_t)|x_t)$ *can be expressed as:*

$$\mathrm{E}(\pi_{t+1}(\bar{\boldsymbol{\kappa}}_t)|x_t) = \beta^K \mathrm{E}(\pi_{t+K+1}(\bar{\boldsymbol{\kappa}}_t)|x_t) + \sum_{h=1}^{K} \beta^{h-1} \bar{\kappa}_{t,t+h} \rho^h x_t, \tag{34}$$

where $\lim_{K\to\infty} \beta^K \mathrm{E}(\pi_{t+K+1}(\bar{\boldsymbol{\kappa}}_t)|x_t)$ *exists. If, moreover, this limit satisfies*

$$\lim_{K\to\infty} \beta^K \mathrm{E}(\pi_{t+K+1}(\bar{\boldsymbol{\kappa}}_t)|x_t) = 0, \tag{35}$$

then $\mathrm{E}(\pi_{t+1}(\bar{\boldsymbol{\kappa}}_t)|x_t)$ *in (34) can be uniquely expressed as:*

$$\mathrm{E}(\pi_{t+1}(\bar{\boldsymbol{\kappa}}_t)|x_t) = \sum_{h=1}^{\infty} \beta^{h-1} \bar{\kappa}_{t,t+h} \rho^h x_t. \tag{36}$$

Proof. Relying on the law of iterated expectations, which holds for the subjective process for $\pi_{t+h}(\bar{\boldsymbol{\kappa}}_t)$, as specified in (33) for fixed t and $h = 1, 2, \ldots$, the subjective conditional expectation $\mathrm{E}(\pi_{t+1}(\bar{\boldsymbol{\kappa}}_t)|x_t)$ can be expressed, by forward iteration, as:

$$\begin{aligned}
\mathrm{E}\left(\pi_{t+1}\left(\bar{\boldsymbol{\kappa}}_t\right)|x_t\right) &= \mathrm{E}\left(\beta E\left(\pi_{t+2}\left(\bar{\boldsymbol{\kappa}}_t\right)|x_{t+1}\right) + \bar{\kappa}_{t,t+1} x_{t+1}\right) \\
&= \beta\,\mathrm{E}\left(\pi_{t+2}\left(\bar{\boldsymbol{\kappa}}_t\right)|x_t\right) + \bar{\kappa}_{t,t+1}\,\mathrm{E}\left(x_{t+1}|x_t\right) \\
&= \beta\,\mathrm{E}\big(\beta E\left(\pi_{t+3}\left(\bar{\boldsymbol{\kappa}}_t\right)|x_{t+2}\right) + \bar{\kappa}_{t,t+2} x_{t+2}|x_t\big) + \bar{\kappa}_{t,t+1}\,\mathrm{E}\left(x_{t+1}|x_t\right) \\
&= \beta^2\,\mathrm{E}\left(\pi_{t+3}\left(\bar{\boldsymbol{\kappa}}_t\right)|x_t\right) + \beta\bar{\kappa}_{t,t+2}\,\mathrm{E}\left(x_{t+2}|x_t\right) + \bar{\kappa}_{t,t+1}\,\mathrm{E}\left(x_{t+1}|x_t\right).
\end{aligned}$$

Continuing the forward iteration, and using that $0 < \rho < 1$ implies that $\mathrm{E}(x_{t+h}|x_t) = \rho^h x_t$, yields:

$$\mathrm{E}\left(\pi_{t+1}\left(\bar{\boldsymbol{\kappa}}_t\right)|x_t\right) = \beta^K\,\mathrm{E}\left(\pi_{t+K+1}\left(\bar{\boldsymbol{\kappa}}_t\right)|x_t\right) + \sum_{h=1}^{K} \beta^{h-1} \bar{\kappa}_{t,t+h} \rho^h x_t. \tag{37}$$

We next show that the last term in (37) converges for $K \to \infty$, which implies that $\lim_{K\to\infty} \mathrm{E}(\pi_{t+K+1}(\bar{\boldsymbol{\kappa}}_t)|x_t)$ exists. Define S_K as:

$$S_K = \sum_{h=1}^{K} \beta^{h-1} \bar{\kappa}_{t,t+h} \rho^h.$$

Because $0 < \kappa_L \leq \bar{\kappa}_{t,t+h} \leq \kappa_U$ for all $h > 0$, $0 < \beta < 1$, and $0 < \rho < 1$, it follows that

$$S_K - S_{K-1} = \beta^{K-1} \bar{\kappa}_{t,t+K} \rho^K > 0,$$

and

$$0 < S_K \leq \sum_{h=1}^{K} \beta^{h-1}\kappa_U\rho^h = \rho\kappa_U \sum_{h=0}^{K-1}(\beta\rho)^h = \frac{(1-(\beta\rho)^K)\rho\kappa_U}{1-\beta\rho}.$$

This shows that S_K is an increasing and bounded sequence and thus it converges for $K \to \infty$. Because the last term in (37) is convergent for $K \to \infty$, it follows that $\beta^K \operatorname{E}\left(\pi_{t+K+1}\left(\bar{\boldsymbol{\kappa}}_t\right)|x_t\right)$ must also be convergent for $K \to \infty$.

If, moreover, we assume that this limit satisfies $\beta^K \operatorname{E}\left(\pi_{t+K+1}\left(\bar{\boldsymbol{\kappa}}_t\right)|x_t\right) \to 0$ for $K \to \infty$, as stated in (35), then the expression in (37) converges, for $K \to \infty$, to:

$$\operatorname{E}\left(\pi_{t+1}\left(\bar{\boldsymbol{\kappa}}_t\right)|x_t\right) = \sum_{h=1}^{\infty} \beta^{h-1}\bar{\kappa}_{t,t+h}\rho^h x_t.$$

■

To show that the representation of participants' inflation expectations in (36) is consistent with the model's predictions of the next period's inflation, we first state the reduced-form expression for inflation. Given the objective and subjective parameters κ_t and $\bar{\boldsymbol{\kappa}}_t$, this is given by:

$$\pi_t(\kappa_t, \bar{\boldsymbol{\kappa}}_t) = \beta\left(\sum_{h=1}^{\infty} \beta^{h-1}\bar{\kappa}_{t,t+h}\rho^h x_t\right) + \kappa_t x_t = \left(\sum_{h=1}^{\infty} \beta^h\rho^h\bar{\kappa}_{t,t+h} + \kappa_t\right)x_t. \tag{38}$$

Using this reduced-form expression for inflation, the following proposition states the set of conditional expectations that constitutes the model's predictions of the next period's inflation. Furthermore, the proposition shows that the representation of participants' inflation expectations, $\operatorname{E}(\pi_{t+1}(\bar{\boldsymbol{\kappa}}_t)|x_t)$ in (36) lies within this set and thus is consistent with the model's predictions.

Proposition 6. *Let $\pi_{t+1}(\kappa_{t+1}, \bar{\boldsymbol{\kappa}}_{t+1})$ denote the reduced-form expression for inflation at time $t+1$, given the objective and subjective parameters κ_{t+1} and $\bar{\boldsymbol{\kappa}}_{t+1}$, as specified in (38). The model's predictions of π_{t+1}, as viewed from time t, are given by the set of conditional expectations:*

$$\begin{aligned}
&\left\{\operatorname{E}\left(\pi_{t+1}\left(\kappa_{t+1}, \bar{\boldsymbol{\kappa}}_{t+1}\right)|x_t\right) \middle| \kappa_{t+1}, \bar{\kappa}_{t+1,t+1+h} \in I^\kappa, h = 1,2,\ldots\right\} \\
&\quad = \left\{\left(\sum_{h=1}^{\infty} \beta^h\rho^h\bar{\kappa}_{t+1,t+1+h} + \kappa_{t+1}\right)\rho x_t \middle| \kappa_{t+1}, \bar{\kappa}_{t+1,t+1+h} \in I^\kappa, h = 1,2,\ldots\right\} \\
&\quad = \begin{cases} \left[\dfrac{\kappa_L\rho}{1-\beta\rho}x_t, \dfrac{\kappa_U\rho}{1-\beta\rho}x_t\right] & \textit{if } x_t \geq 0, \\ \left[\dfrac{\kappa_U\rho}{1-\beta\rho}x_t, \dfrac{\kappa_L\rho}{1-\beta\rho}x_t\right] & \textit{if } x_t < 0. \end{cases}
\end{aligned} \tag{39}$$

This set of conditional expectations formalizes the Knightian uncertainty about π_{t+1}.

Moreover, the representation of participants' inflation expectations, $\mathrm{E}\left(\pi_{t+1}\left(\bar{\boldsymbol{\kappa}}_t\right)|x_t\right)$ *in (36), is consistent with the model's predictions, in the sense that*

$$\mathrm{E}\left(\pi_{t+1}\left(\bar{\boldsymbol{\kappa}}_t\right)|x_t\right) \in \left\{ \mathrm{E}\left(\pi_{t+1}\left(\kappa_{t+1}, \bar{\boldsymbol{\kappa}}_{t+1}\right)|x_t\right) \middle| \kappa_{t+1}, \bar{\kappa}_{t+1,t+1+h} \in I^{\kappa}, h = 1, 2, \dots \right\},$$

for any value of the subjective parameters $\bar{\kappa}_{t,t+h} \in I^{\kappa}$ *for* $h = 1, 2, \dots$.

Proof. The reduced-form expression for $\pi_{t+1}(\kappa_{t+1}, \bar{\boldsymbol{\kappa}}_{t+1})$, given κ_{t+1} and $\bar{\boldsymbol{\kappa}}_{t+1}$, is given by:

$$\pi_{t+1}\left(\kappa_{t+1}, \bar{\boldsymbol{\kappa}}_{t+1}\right) = \left(\sum_{h=1}^{\infty} \beta^h \rho^h \bar{\kappa}_{t+1,t+1+h} + \kappa_{t+1}\right) x_{t+1}.$$

which implies the the following expression for the conditional expectation of the next period's inflation, given κ_{t+1} and $\bar{\boldsymbol{\kappa}}_{t+1}$,

$$\begin{aligned}
\mathrm{E}\left(\pi_{t+1}\left(\kappa_{t+1}, \bar{\boldsymbol{\kappa}}_{t+1}\right)|x_t\right) &= \mathrm{E}\left(\left(\sum_{h=1}^{\infty} \beta^h \rho^h \bar{\kappa}_{t+1,t+1+h} + \kappa_{t+1}\right) x_{t+1} \middle| x_t\right) \\
&= \left(\sum_{h=1}^{\infty} \beta^h \rho^h \bar{\kappa}_{t+1,t+1+h} + \kappa_{t+1}\right) \mathrm{E}\left(x_{t+1}|x_t\right) \\
&= \left(\sum_{h=1}^{\infty} \beta^h \rho^h \bar{\kappa}_{t+1,t+1+h} + \kappa_{t+1}\right) \rho x_t.
\end{aligned}$$

Here, $0 < \beta < 1$ and $0 < \rho < 1$ imply that $\mathrm{E}\left(\pi_{t+1}\left(\kappa_{t+1}, \bar{\boldsymbol{\kappa}}_{t+1}\right)|x_t\right)$ is increasing in κ_{t+1} and $\bar{\kappa}_{t+1,t+1+h}$ for all $h > 0$. Moreover, as $0 < \kappa_L \le \kappa_{t+1} \le \kappa_U$ and $0 < \kappa_L \le \bar{\kappa}_{t+1,t+1+h} \le \kappa_U$ for all $h = 1, 2, \dots$, it follows that

$$\sum_{h=1}^{\infty} \beta^h \rho^h \bar{\kappa}_{t+1,t+1+h} + \kappa_{t+1} \ge \sum_{h=1}^{\infty} \beta^h \rho^h \kappa_L + \kappa_L = \frac{\beta\rho\kappa_L}{1-\beta\rho} + \kappa_L = \frac{\kappa_L}{1-\beta\rho},$$

and

$$\sum_{h=1}^{\infty} \beta^h \rho^h \bar{\kappa}_{t+1,t+1+h} + \kappa_{t+1} \le \sum_{h=1}^{\infty} \beta^h \rho^h \kappa_U + \kappa_U = \frac{\beta\rho\kappa_U}{1-\beta\rho} + \kappa_U = \frac{\kappa_U}{1-\beta\rho}.$$

Thus, it follows that the set of conditional expectations of $\pi_{t+1}(\kappa_{t+1}, \bar{\boldsymbol{\kappa}}_{t+1})$ for all possible values of $\kappa_{t+1}, \bar{\kappa}_{t+1,t+1+h} \in I^\kappa = [\kappa_L, \kappa_U]$ for $h = 1, 2, \dots$ is given by the set:

$$\left\{ \mathrm{E}\left(\pi_{t+1}\left(\kappa_{t+1}, \bar{\boldsymbol{\kappa}}_{t+1}\right) | x_t\right) \middle| \kappa_{t+1}, \bar{\kappa}_{t+1,t+1+h} \in I^\kappa, h = 1, 2, \dots \right\}$$
$$= \left\{ \left(\sum_{h=1}^{\infty} \beta^h \rho^h \bar{\kappa}_{t+1,t+1+h} + \kappa_{t+1} \right) \rho x_t \middle| \kappa_{t+1}, \bar{\kappa}_{t+1,t+1+h} \in I^\kappa, h = 1, 2, \dots \right\}$$
$$= \begin{cases} \left[\dfrac{\kappa_L \rho}{1 - \beta\rho} x_t, \dfrac{\kappa_U \rho}{1 - \beta\rho} x_t \right] & \text{if } x_t \geq 0, \\ \left[\dfrac{\kappa_U \rho}{1 - \beta\rho} x_t, \dfrac{\kappa_L \rho}{1 - \beta\rho} x_t \right] & \text{if } x_t < 0. \end{cases}$$

Moreover, it follows from $\bar{\kappa}_{t,t+h} \in I^\kappa = [\kappa_L, \kappa_U]$ for $h = 1, 2, \dots$ that the infinite sum $\sum_{h=1}^{\infty} \beta^{h-1} \bar{\kappa}_{t,t+h} \rho^h$ in $\mathrm{E}(\pi_{t+1}(\bar{\boldsymbol{\kappa}}_t)|x_t) = \sum_{h=1}^{\infty} \beta^{h-1} \bar{\kappa}_{t,t+h} \rho^h x_t$ in (6) satisfies

$$\sum_{h=1}^{\infty} \beta^{h-1} \bar{\kappa}_{t,t+h} \rho^h \geq \sum_{h=1}^{\infty} \beta^{h-1} \kappa_L \rho^h = \rho\kappa_L \sum_{h=0}^{\infty} (\beta\rho)^h = \frac{\rho\kappa_L}{1 - \beta\rho},$$

and

$$\sum_{h=1}^{\infty} \beta^{h-1} \bar{\kappa}_{t,t+h} \rho^h \leq \sum_{h=1}^{\infty} \beta^{h-1} \kappa_L \rho^h = \rho\kappa_L \sum_{h=0}^{\infty} (\beta\rho)^h = \frac{\rho\kappa_L}{1 - \beta\rho},$$

which implies that participants' subjective inflation expectation, $\mathrm{E}(\pi_{t+1}(\bar{\boldsymbol{\kappa}}_t)|x_t)$ in (36), lies within the set of conditional expectations of $\pi_{t+1}(\kappa_{t+1}, \bar{\boldsymbol{\kappa}}_{t+1})$ in (39). ■